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On weakly para-cosymplectic manifolds of dimension 3

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Abstract

The local structure of a 3-dimensional essentially weakly para-cosymplectic manifold is described in two ways: using special adapted local frames and special coordinate systems. This enables a description of the curvature of such manifolds. Local isometries and Killing vector fields are also investigated. It is proved that if a 3-dimensional weakly para-cosymplectic manifold is locally homogeneous as a Riemannian manifold, then it is para-cosymplectic or locally flat. Then a classification of such manifolds is given.

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1. Introduction

Let M be a $(2n+1)$-dimensional connected differentiable manifold. Denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. From now on, $X, Y, Z, \dots$ denote arbitrary smooth vector fields on M, i.e. elements of $\mathfrak{X}(M)$.

An almost para-contact hyperbolic metric structure on M is a quadruple (φ, ξ, η, g) consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g on M satisfying the following relations [5]:

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y).$$

As consequences of the above, we additionally have $\varphi\xi = 0$, $\eta(\varphi X) = 0$, $\eta(X) = g(X,\xi)$, $(\xi,\xi) = 1$, $g(\varphi X, Y) + g(\varphi Y, X) = 0$. Thus, Φ defined by $\Phi(X,Y) = g(\varphi X, Y)$, is a (skew-symmetric) 2-form on M, which is called the fundamental form (of the structure).

The manifold M endowed with an almost para-contact hyperbolic metric structure (φ, ξ, η, g) will be called (cf. [4])

(a) para-cosymplectic if η and Φ are parallel with respect to the Levi-Civita connection ∇ of the metric g ($\nabla\eta = 0$, $\nabla\Phi = 0$);

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(b) almost para-cosymplectic if the forms η and Φ are closed ($d\eta = 0$, $d\Phi = 0$);

(c) weakly para-cosymplectic if it is almost para-cosymplectic and its curvature operators $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ commute with φ, that is,

$$[R(X,Y),\varphi] = R(X,Y)\varphi - \varphi R(X,Y) = 0. \tag{1}$$

We have the following strict inclusions for these classes of manifolds: para-cosymplectic $\subset$ weakly para-cosymplectic $\subset$ almost para-cosymplectic.

The above notions are para-contact with hyperbolic metric analogues of (almost) cosymplectic manifolds (for cosymplectic and almost cosymplectic manifolds, see [1,7,8], etc.).

Our definition of the para-cosymplecticity differs from that used in the paper [5], in which this notion concerns even-dimensional indefinite almost Hermitian or almost para-Hermitian manifolds with coclosed fundamental forms.

Let M be an almost para-cosymplectic manifold. Then $\mathcal{D} = \ker\eta$ is a $2n$-dimensional involutive distribution on M. Let $\mathcal{F}$ be the foliation of M generated by $\mathcal{D}$. Then $\mathcal{D}$ and $\mathcal{F}$ are called, respectively, the canonical distribution and the canonical foliation of M. Moreover, $\mathcal{D}$ is φ-invariant since $\mathcal{D} = \operatorname{Im}\varphi$. Therefore, on any leaf $\bar{M}$ of $\mathcal{F}$, the restrictions $J = \varphi|_{\bar{M}}$ and $G = g|_{\bar{M}}$ define an almost para-Hermitian structure on $\bar{M}$ (see [2,3] for such structures). This means that $J^2\bar{X} = \bar{X}$ and $G(J\bar{X}, J\bar{Y}) = -G(\bar{X},\bar{Y})$ for any vector fields $\bar{X},\bar{Y} \in \mathfrak{X}(\bar{M})$. The fundamental form Ω of the structure (J,G), $\Omega(\bar{X},\bar{Y}) = G(J\bar{X},\bar{Y})$, is the pull-back of Φ, so it is closed. Thus, $\bar{M}$ equipped with (J,G) becomes an almost para-Kählerian manifold.

Let A be the (1, 1)-tensor field on M defined by

$$AX = -\nabla_X\xi. \tag{2}$$

One notes that A restricted to a leaf $\bar{M}$ of $\mathcal{F}$ is just the shape operator of $\bar{M}$. Algebraic properties of A can be listed as follows (cf. [4]):

$$\begin{aligned} &g(AX,Y) = g(AY,X), \quad A\xi = 0, \quad \eta\circ A = 0, \quad A\varphi + \varphi A = 0,\\ &g(\varphi AX,Y) = g(\varphi AY,X), \quad \operatorname{Trace}(\varphi A) = \operatorname{Trace}(A) = 0. \end{aligned} \tag{3}$$

If for any leaf $\bar{M}$ of the canonical foliation $\mathcal{F}$, the structure (J,G) induced on $\bar{M}$ is para-Kählerian ($\bar{\nabla}J = 0$), we say that the almost para-cosymplectic manifold M has para-Kählerian leaves. An almost para-cosymplectic manifold has para-Kählerian leaves if and only if (see [4])

$$N_\varphi(X,Y) = 2\eta(X)AY - 2\eta(Y)AX, \tag{4}$$

or, equivalently,

$$(\nabla_X\varphi)Y = g(A\varphi X,Y)\xi - \eta(Y)A\varphi X, \tag{5}$$

where N_φ is the Nijenhuis torsion tensor corresponding to φ,

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y].$$

The rest of this paper is devoted to studying weakly para-cosymplectic manifolds in dimension 3.

2. Local structure frames

Since any 3-dimensional almost para-cosymplectic manifold has para-Kählerian leaves, Theorem 5 from [4] enables us to claim that a 3-dimensional almost para-cosymplectic manifold is weakly para-cosymplectic if and only if the following two conditions are fulfilled:

(I) A is a Codazzi tensor field, that is,

$$(\nabla_XA)Y - (\nabla_YA)X = 0; \tag{6}$$

(II) at any point $p \in M$, either (a) $A_p = 0$ or (b) there exists a non-zero null vector $v \in T_pM$ such that $A_pu = \varepsilon_1 g(u,v)v$ for any $u \in T_pM$ and $\varphi v = \varepsilon_2 v$, where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

A para-cosymplectic manifold is locally a product of an open interval and a para-Kählerian manifold (cf. [4]). In this context, we will concentrate our study on the case when $\nabla\varphi \neq 0$.

If $\nabla\varphi \neq 0$ at every point of a weakly para-cosymplectic manifold M, then we say that M is essentially weakly para-cosymplectic.

A 3-dimensional almost para-cosymplectic manifold is essentially weakly para-cosymplectic if and only if the conditions (I) and (II)(b) hold. Indeed, by virtue of (5), one claims that at a point of a 3-dimensional almost para-cosymplectic manifold the following equivalence holds: $\nabla\varphi = 0$ if and only if $A = 0$.

Let M be a 3-dimensional essentially weakly para-cosymplectic manifold.

Lemma 1. *For any $p \in M$, there exist a neighborhood $\mathcal{O}$ of p and a vector field $V \in \mathfrak{X}(\mathcal{O})$, which is unique up to a sign, non-zero, null, orthogonal to ξ and such that*

$$AX = \varepsilon_1 g(X, V)V \tag{7}$$

for any $X \in \mathfrak{X}(\mathcal{O})$, where $\varepsilon_1 = \pm 1$. Moreover, $\varphi V = \varepsilon_2 V$ with $\varepsilon_2 = \pm 1$.

Proof. Fix an arbitrary point $p \in M$. Next choose an open connected neighborhood $\mathcal{O}$ of p, a vector field $Z \in \mathfrak{X}(\mathcal{O})$ such that $g(AZ, Z) \neq 0$ at every point of $\mathcal{O}$. Normalizing Z, we assume that $g(AZ, Z) = \varepsilon_1$, where $|\varepsilon_1| = 1$. We will show that $V = AZ$ is just the desired vector field. For, if $X \in \mathfrak{X}(\mathcal{O})$, then there exists a smooth function $f \in \mathcal{F}(\mathcal{O})$ for which $AX = fV$. Hence $g(AX, Z) = fg(V, Z) = fg(AZ, Z) = \varepsilon_1 f$. Consequently, $f = \varepsilon_1 g(AX, Z) = \varepsilon_1 g(X, AZ) = \varepsilon_1 g(X, V)$. This gives (7). Using the anticommutativity $A\varphi + \varphi A = 0$ (cf. (3)) and (7), we have $g(\varphi X, V)V + g(X, V)\varphi V = 0$, which shows that φV and V are collinear. Hence $\varphi V = \varepsilon_2 V$ for a certain $\varepsilon_2 = \pm 1$. The rest of the assertion is an obvious consequence of (II)(b). □

Lemma 2. *For any $p \in M$, there exist a neighborhood $\mathcal{O}$ of p and a frame of vector fields (V, U, ξ) on $\mathcal{O}$ such that*

$$g(V, U) = g(\xi, \xi) = 1, \quad g(V, V) = g(U, U) = g(V, \xi) = g(U, \xi) = 0. \tag{8}$$

With respect to (V, U, ξ), the linear operators φ and A are given by

$$\begin{aligned} &\text{(a)} \quad AV = 0, \quad AU = \varepsilon_1 V, \quad A\xi = 0,\\ &\text{(b)} \quad \varphi V = \varepsilon_2 V, \quad \varphi U = -\varepsilon_2 U, \quad \varphi\xi = 0. \end{aligned} \tag{9}$$

Consequently, the second fundamental form Φ is given on $\mathcal{O}$ by

$$\Phi(V, U) = \varepsilon_2, \quad \Phi(V, \xi) = \Phi(U, \xi) = 0. \tag{10}$$

Proof. We use the neighborhood $\mathcal{O}$ and the vector field V from Lemma 1 and additionally choose uniquely the vector field $U \in \mathfrak{X}(\mathcal{O})$ such that $\varphi U = -\varepsilon_2 U$ and $g(U, V) = 1$. Since U is an eigenvector field of φ, $g(U, U) = 0$. The rest of the relations can be checked using (7). □

Lemma 3. *With respect to the structure frame (V, U, ξ), the Levi-Civita connection ∇ of M is given by*

$$\begin{aligned} &\text{(a)} \quad \nabla_V V = \nabla_\xi V = 0, \quad \nabla_U V = \beta V,\\ &\text{(b)} \quad \nabla_V U = \nabla_\xi U = 0, \quad \nabla_U U = \varepsilon_1 \xi - \beta U,\\ &\text{(c)} \quad \nabla_V \xi = \nabla_\xi \xi = 0, \quad \nabla_U \xi = -\varepsilon_1 V, \end{aligned} \tag{11}$$

β being a certain function on $\mathcal{O}$. Consequently, for the Lie brackets of V, U, ξ, we have

$$[V, U] = -\beta V, \quad [V, \xi] = 0, \quad [U, \xi] = -\varepsilon_1 V, \tag{12}$$

and $\mathrm{d}\beta(\xi) = 0$.

Proof. Consider any $X, Y \in \mathfrak{X}(\mathcal{O})$. First, we prove that the covariant derivatives of V are given by

$$\nabla_X V = \beta g(X, V)V \tag{13}$$

for a certain function β on $\mathcal{O}$. For, using formulas (7) and (8), we find that $g(\nabla_X V, V) = 0$ and

$$g(\nabla_X V, \xi) = -g(V, \nabla_X \xi) = g(V, AX) = \varepsilon_1 g(X, V)g(V, V) = 0.$$

Therefore, $\nabla_X V$ is collinear with V, so we can write

$$\nabla_X V = \tau(X)V, \tag{14}$$

where τ is a 1-form on $\mathcal{O}$. Now, using (7) and (14), we get

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X AY - A\nabla_X Y \\ &= \varepsilon_1 g(Y, \nabla_X V)V + \varepsilon_1 g(Y, V)\nabla_X V = 2\varepsilon_1 \tau(X) g(Y, V)V.\end{aligned} \tag{15}$$

Recalling the Codazzi condition (6) and using (15), we obtain $\tau(X)g(Y, V) = \tau(Y)g(X, V)$, which implies

$$\tau(X) = \beta g(X, V) \tag{16}$$

for a certain function β on $\mathcal{O}$. Finally, (16) applied to (14) gives (13).

The formulas (11)(a) follow now from (13) if we use (8). By (2), $\nabla_X \xi = -AX$; therefore the formulas (11)(c) are in fact consequences of (9)(a). Moreover, using (8), the already obtained formulas (11)(a), (c) and the metricity of ∇, we find (11)(b). Now, (12) follows from (11). Finally, $\mathrm{d}\beta(\xi) = 0$ is a consequence of (12) and the Jacobi identity

$$[[V, U], \xi] + [[U, \xi], V] + [[\xi, V], U] = 0,$$

which completes the proof. □

In what follows, the frame (V, U, ξ) and the function β that appeared in the above lemmas will be called the structure frame and the structure function on the set $\mathcal{O}$.

Proposition 4. *The Riemann curvature operators $R(X, Y)$, the Ricci curvature tensor S and the scalar curvature r of a 3-dimensional essentially weakly para-cosymplectic manifold M are given by*

$$R(X, Y) = (r/2)\Phi(X, Y)\varphi, \tag{17}$$

$$S(X, Y) = (r/2)(g(X, Y) - \eta(X)\eta(Y)), \tag{18}$$

$$r = 2\mathrm{d}\beta(V), \tag{19}$$

Φ being the second fundamental form of M.

Proof. First note that

$$R(X, Y)\xi = 0 \tag{20}$$

is a consequence of $[R(X, Y), \varphi]\xi = 0$. Moreover, using (14) we can calculate the following:

$$\begin{aligned}R(X, Y)V &= \nabla^2_{X,Y} V - \nabla^2_{Y,X} V = ((\nabla_X \tau)(Y) - (\nabla_Y \tau)(X))\, V \\ &= 2\mathrm{d}\tau(X, Y)V.\end{aligned} \tag{21}$$

Additionally, the algebraic properties of the Riemann curvature tensor R and (20), (21) imply

$$R(X, Y)U = -2\mathrm{d}\tau(X, Y)U. \tag{22}$$

Comparing (20)–(22) with (9)(b), we conclude that

$$R(X, Y) = 2\varepsilon_2 \mathrm{d}\tau(X, Y)\varphi. \tag{23}$$

Eq. (23) together with (9)(b) applied to the first Bianchi identity gives

$$0 = R(\xi, V)U + R(V, U)\xi + R(U, \xi)V = -2\mathrm{d}\tau(\xi, V)U + 2\mathrm{d}\tau(U, \xi)V.$$

This leads to $\mathrm{d}\tau(\xi, V) = \mathrm{d}\tau(\xi, U) = 0$, which compared to (10) shows that the 2-forms $\mathrm{d}\tau$ and Φ are collinear; to be precise,

$$\mathrm{d}\tau(X, Y) = \varepsilon_2 \mathrm{d}\tau(V, U)\Phi(X, Y). \tag{24}$$

On the other hand, using (16) and (14), we find

$$\mathrm{d}\tau(X, Y) = \frac{1}{2}\left(\mathrm{d}\beta(X)g(Y, V) - \mathrm{d}\beta(Y)g(X, V)\right).$$

Hence using also (8), we get $\mathrm{d}\tau(V, U) = (1/2)\mathrm{d}\beta(V)$. This together with (24) applied to (23) yields

$$R(X, Y) = \mathrm{d}\beta(V)\, \Phi(X, Y)\, \varphi. \tag{25}$$

Therefore for the Ricci tensor, we obtain

$$S(Y, Z) = \mathrm{d}\beta(V)(g(Y, Z) - \eta(Y)\eta(Z)), \tag{26}$$

and hence for the scalar curvature, we get (19). Finally, (17) and (18) follow from (25) and (26) in view of (19). □

3. Local structure

Now, we are going to prove theorems characterizing locally essentially weakly para-cosymplectic manifolds among 3-dimensional almost para-contact manifolds with a hyperbolic metric.

Proposition 5. *For a 3-dimensional almost para-contact manifold with a hyperbolic metric, the following conditions are equivalent:*

(i) *M is essentially weakly para-cosymplectic.*
(ii) *For any point $p \in M$, there exist a neighborhood $\mathcal{O}$ of p and vector fields $V, U \in \mathfrak{X}(\mathcal{O})$ such that*

$$\begin{aligned} &[V, U] = -\beta V, \quad [V, \xi] = 0, \quad [U, \xi] = -\varepsilon_1 V, \\ &g(V, U) = g(\xi, \xi) = 1, \quad g(V, V) = g(U, U) = g(V, \xi) = g(U, \xi) = 0, \\ &\varphi V = \varepsilon_2 V, \quad \varphi U = -\varepsilon_2 U, \quad \varphi\xi = 0, \quad \eta(\cdot) = g(\cdot, \xi), \end{aligned} \tag{27}$$

where $|\varepsilon_1| = |\varepsilon_2| = 1$ and β is a function on $\mathcal{O}$.

Proof. The implication (i) ⇒ (ii) follows from Lemmas 2 and 3.

(ii) ⇒ (i). Let a 3-dimensional almost para-contact hyperbolic metric manifold M satisfy (27). First, we note that under this assumption, for the fundamental form Φ of M, we must have $\Phi(V, U) = \varepsilon_2$ and $\Phi(V, \xi) = \Phi(U, \xi) = 0$. Therefore, for the exterior derivative of Φ, it is possible to obtain

$$3\mathrm{d}\Phi(V, U, \xi) = V\,\Phi(U, \xi) + U\,\Phi(\xi, V) + \xi\,\Phi(V, U) - \Phi([V, U], \xi) - \Phi([U, \xi], V) - \Phi([\xi, V], U) = 0.$$

Additionally, since the exterior derivative of the form η can be expressed as

$$2\mathrm{d}\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]),$$

a computation shows that $\mathrm{d}\eta(V, U) = \mathrm{d}\eta(V, \xi) = \mathrm{d}\eta(U, \xi) = 0$. Thus, the forms Φ and η are closed and the manifold M is almost para-cosymplectic.

Furthermore, by the assumption (27), the Levi-Civita connection of M is given just by the formulas (11). And it is also clear that the curvature of M is given by the formula (17) because it is in fact a consequence of the expression of the Levi-Civita connection (11) only. Now, after using (17), the formula (1) follows easily, so M is weakly para-cosymplectic.

It remains to prove that $\nabla\varphi \neq 0$ at every point of M. But applying (11), we compute $(\nabla_U \Phi)(U, \xi) = -\varepsilon_1\varepsilon_2 \neq 0$, which completes the proof. □

Proposition 6. *For a 3-dimensional almost para-contact hyperbolic metric manifold, the following conditions are equivalent:*

(i) *M is essentially weakly para-cosymplectic.*
(ii) *For any $p \in M$, there exists a local chart $(\mathcal{O}, (x, y, z))$ centred at p and such that*

$$\begin{aligned} &\xi = \partial_z, \quad \eta = \mathrm{d}z, \\ &\varphi\partial_x = \varepsilon_2\partial_x, \quad \varphi\partial_y = 2\varepsilon_2(b(x, y) - \varepsilon_1 z)\partial_x - \varepsilon_2\partial_y, \quad \varphi\partial_z = 0, \\ &g(\partial_x, \partial_y) = g(\partial_z, \partial_z) = 1, \quad g(\partial_y, \partial_y) = 2(b(x, y) - \varepsilon_1 z), \\ &g(\partial_x, \partial_x) = g(\partial_x, \partial_z) = g(\partial_y, \partial_z) = 0, \end{aligned} \tag{28}$$

where $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $\partial_z = \partial/\partial z$, b is a function depending on two variables only and $\varepsilon_1, \varepsilon_2$ are real constants such that $|\varepsilon_1| = |\varepsilon_2| = 1$.

Proof. (i) $\Rightarrow$ (ii). We are going to apply Proposition 5. First we find expressions for the frame vectors V, U, ξ in a certain special coordinate system.

By $[V, \xi] = 0$, the vector fields V and ξ are straightened simultaneously on a coordinate neighborhood $(\mathcal{O}', (x', y', z'))$ of p. So we have

$$V = \partial_{x'}, \quad U = u_{x'}\partial_{x'} + u_{y'}\partial_{y'} + u_{z'}\partial_{z'}, \quad \xi = \partial_{z'}$$

for certain functions $u_{x'}, u_{y'}, u_{z'}$ with $u_{y'} \neq 0$ on $\mathcal{O}'$. For simplicity, we can assume that $\mathcal{O}'$ is a cubical neighborhood of p, say $\mathcal{O}' = P^3$, where $P = (-a, a)$ and a is a positive real number. Furthermore, as regards the commutators (12), we have

$$\partial_{x'}u_{x'} = -\beta, \quad \partial_{z'}u_{x'} = \varepsilon_1, \quad \partial_{x'}u_{y'} = \partial_{z'}u_{y'} = \partial_{x'}u_{z'} = \partial_{z'}u_{z'} = 0.$$

From the above system of equations, it follows that

$$u_{x'} = h(x', y') + \varepsilon_1 z', \quad u_{y'} = u_{y'}(y'), \quad u_{z'} = u_{z'}(y')$$

for a certain function h of two variables.

Let $v\colon P \to \mathbb{R}$ and $w\colon P \to \mathbb{R}$ be functions defined by

$$v(t) = \int_0^t \frac{\mathrm{d}s}{u_{y'}(s)}, \quad w(t) = \int_0^t \frac{u_{z'}(s)}{u_{y'}(s)}\mathrm{d}s, \quad t \in P.$$

The function v has the inverse $v^{-1}\colon v(P) \to P$. We introduce new coordinates (x, y, z), $x, z \in P$, $y \in v(P)$ on a neighborhood $\mathcal{O}$ of p by assuming

$$x' = x, \quad y' = v^{-1}(y), \quad z' = z + \widetilde{w}(y),$$

where $\widetilde{w}\colon v(P) \to \mathbb{R}$ stands for the composition $w \circ v^{-1}$. One verifies that

$$\begin{aligned} V &= \partial_{x'} = \partial_x, \quad \xi = \partial_{z'} = \partial_z, \\ U &= u_{y'}(y')\partial_{y'} + \left(\varepsilon_1 z' + h(x', y')\right)\partial_{x'} + u_{z'}(y')\partial_{y'} \\ &= (\varepsilon_1 z + \varepsilon_1 \widetilde{w}(y)) + h(x, v^{-1}(y))\partial_x + \partial_y. \end{aligned}$$

Denoting the term $\varepsilon_1 \widetilde{w}(y) + h(x, v^{-1}(y))$ simply by $-b(x, y)$, we have

$$V = \partial_x, \quad U = (\varepsilon_1 z - b(x, y))\partial_x + \partial_y, \quad \xi = \partial_z. \tag{29}$$

Therefore, we can write on $\mathcal{O}$,

$$\partial_x = V, \quad \partial_y = (b(x, y) - \varepsilon_1 z)V + U, \quad \partial_z = \xi.$$

Now, to obtain (28), it is sufficient to use (8) and (9)(b).

(ii) $\Rightarrow$ (i). Conversely, if we assume (28) and define a local frame (V, U, ξ) like in the formula (29), then it is straightforward to verify that (27) holds. This in view of Proposition 5 completes the proof. □

In the sequel, the local charts $(\mathcal{O}, (x, y, z))$ constructed in Proposition 6 will be called canonical charts of an essentially weakly para-cosymplectic manifold.

Lemma 7. *With respect to a canonical chart* $(\mathcal{O}, (x, y, z))$ *of a* 3*-dimensional essentially weakly para-cosymplectic manifold* M*, we have*

$$\beta = \partial_x b, \qquad r = 2\partial_x^2 b. \tag{30}$$

Proof. Recalling (29), we see that $[V, U] = -(\partial_x b)V$. This compared to $[V, U] = -\beta V$ (cf. (12)) leads to the first equality of (30). The second equality of (30) follows from the first one and (19). □

4. Local isometries and Killing vector fields

Proposition 8. *Let M be a 3-dimensional essentially weakly para-cosymplectic manifold, whose curvature does not vanish at any point. If $p \in M$ and $f : \mathcal{O} \to f(\mathcal{O}) \subset M$ is a local isometry, then f is a local isomorphism of the structure (φ, ξ, η, g), that is,*

$$f^* g = g, \quad f_* \xi = \xi, \quad f^* \Phi = \Phi, \quad f_* \circ \varphi = \varphi \circ f_*, \quad f_* \eta = \eta.$$

Moreover, for a certain $\varepsilon = \pm 1$,

$$f_* V = \varepsilon V, \quad f_* U = \varepsilon U, \quad \beta \circ f = \varepsilon \beta.$$

Proof. By (17), for the Riemann curvature tensor of M, we have the formula

$$R(X, Y, Z, W) = g(R(X, Y)Z, W) = \frac{r}{2} \Phi(X, Y)\Phi(Z, W).$$

On the other hand, when f is a local isometry, then

$$R(f_* X, f_* Y, f_* Z, f_* W) = R(X, Y, Z, W), \quad r \circ f = r.$$

Hence, for the fundamental form Φ, we must have $f^* \Phi = \alpha \Phi$, $\alpha = \pm 1$, and moreover $f_* \circ \varphi = \alpha \varphi \circ f_*$.

Since $\varphi \xi = 0$, it must be that $\varphi f_* \xi = \alpha f_* \varphi \xi = 0$, so $f_* \xi = \alpha_1 \xi$, α_1 being a function. But since ξ and $f_* \xi$ are unit vector fields, $\alpha_1 = \pm 1$. Moreover, using also (9)(b), we find $\varphi f_* V = \alpha f_* \varphi V = \alpha \varepsilon_2 f_* V$ and $\varphi f_* U = \alpha f_* \varphi U = -\alpha \varepsilon_2 f_* U$. This means that $f_* V$ and $f_* U$ are eigenvector fields for φ and they correspond to different eigenvalues. Therefore, one of them is collinear with V and the other one with U.

Having $AX = -\nabla_X \xi$, we can find in general the following:

$$f_* AX = -f_* \nabla_X \xi = -\nabla_{f_* X} f_* \xi = -\alpha_1 \nabla_{f_* X} \xi = \alpha_1 A f_* X. \tag{31}$$

Using (31) and (9)(a), we get $\alpha_1 A f_* V = f_* AV = 0$. Thus $f_* V$ must be collinear with V, and $f_* U$ must be collinear with U; say $f_* V = \alpha_2 V$ and $f_* U = \alpha_3 U$. Thus $g(f_* V, f_* U) = g(V, U) = 1$ implies now $\alpha_2 \alpha_3 = 1$. Additionally, using $\Phi(f_* V, f_* U) = \alpha \Phi(V, U)$, we have $\alpha_2 \alpha_3 = \alpha$. Hence, $\alpha = 1$.

From (31) it also follows that $f_* AU = \alpha_1 A f_* U$. Hence, using also (9)(a), we obtain $\alpha_2 = \alpha_1 \alpha_3$. This, compared to $\alpha_2 \alpha_3 = 1$, gives $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \pm 1$. Finally, by (11)(a) and $f_* \nabla_U V = \nabla_{f_* U} f_* V$, we have $\alpha_2 \beta = \beta \circ f$. This completes the proof. □

The "infinitesimal" version of Proposition 8 can be formulated as follows:

Proposition 9. *Let M be a 3-dimensional essentially weakly para-cosymplectic manifold, whose curvature does not vanish at any point. A vector field K on M is Killing if and only if it commutes with the vector fields V, U, ξ, that is, $[K, \xi] = [K, V] = [K, U] = 0$. Consequently, for a Killing vector field on M, it holds that $K\beta = 0$.*

Proof. Let $p \in M$ and f_t, $-\alpha < t < \alpha$, $\alpha > 0$, be the local flow of isometries generated by K around p. Then by Proposition 8,

$$(f_t)_* \xi = \xi, \quad (f_t)_* V = \varepsilon(t) V, \quad (f_t)_* U = \varepsilon(t) U,$$

where $\varepsilon(t) = \pm 1$. In fact $\varepsilon(t) = 1$ since it is a smooth function of t and $\varepsilon(0) = 1$. Therefore, using the famous formula for the Lie brackets of vector fields, we obtain

$$[K, V]_p = \lim_{t \to 0} \frac{(f_{-t})_* V_{f_t(p)} - V_p}{t} = 0.$$

Hence $[K, V] = 0$ on M. Similarly, $[K, U] = [K, \xi] = 0$ on M. The equality $\beta = \beta \circ f_t$ shows that β is constant along the trajectories of K; therefore $K\beta = 0$. □

Proposition 10. *Let M be a 3-dimensional essentially weakly para-cosymplectic manifold whose curvature does not vanish at any point. Let K be a non-zero vector field on M. K is Killing if and only if for any point $p \in M$ and any canonical chart $(\mathcal{O}, (x, y, z))$ centred at p, there exists an open set $\mathcal{O}'$ such that $p \in \mathcal{O}' \subset \mathcal{O}$ and*

$$b(x, y) = v(x - u(y)) - u'(y) + cy, \tag{32}$$

$$K = a(u'(y)\partial_x + \partial_y + \varepsilon_1 c\partial_z) \tag{33}$$

on $\mathcal{O}'$, where u and v are certain functions of one variable and $a, c =$ const., $c \neq 0$.

Proof. Let K be a non-zero Killing vector field on M. By Proposition 9, we know that

$$[K, V] = [K, U] = [K, \xi] = 0. \tag{34}$$

First, we note that $K \neq 0$ at every point of M. In fact, if $K_q = 0$ at a certain point $q \in M$, then using (34) we have $\nabla_{V_q} K = [V, K]_q + \nabla_{K_q} V = 0$, and similarly $\nabla_{U_q} K = \nabla_{\xi_q} K = 0$. Therefore, $(\nabla K)_q = 0$. Since M is connected, $K = 0$ on M, which is a contradiction.

Let $p \in M$ and $(\mathcal{O}, (x, y, z))$ be a canonical chart centred at p. We write on $\mathcal{O}$: $K = K_x\partial_x + K_y\partial_y + K_z\partial_z$. Using (29), we check that conditions (34) imply the following for the components K_x, K_y, K_z:

$$\begin{aligned} &\text{(a)} \quad K_x = K_x(y), \quad K_y = \text{const.}, \quad K_z = \text{const.}, \\ &\text{(b)} \quad \partial_y K_x + K_x\partial_x b + K_y\partial_y b - \varepsilon_1 K_z = 0. \end{aligned} \tag{35}$$

Note that additionally $K_y \neq 0$. For, let $K_y = 0$. Then from (35) it follows that $K_x\partial_x^2 b = 0$. This together with (30) leads to $rK_x = 0$. Since $r \neq 0$, $K_x = 0$ on $\mathcal{O}$. But then also $K_z = 0$ on $\mathcal{O}$, which is impossible as $K \neq 0$ everywhere on M.

We suppose

$$a = K_y, \quad w(y) = K_x(y)/a, \quad c = \varepsilon_1 K_z/a, \quad \tilde{b}(x, y) = b(x, y) - cy.$$

In view of the above and (35)(b), we have the following:

$$\partial_y(w(y) + \tilde{b}(x, y)) + \partial_x(w(y)\tilde{b}(x, y)) = 0.$$

Therefore, in a neighborhood of p there exists a function $h(x, y)$ such that

$$\partial_x h(x, y) = w(y) + \tilde{b}(x, y), \quad \partial_y h(x, y) = -w(y)\tilde{b}(x, y). \tag{36}$$

Next, in a neighborhood of p, we change the coordinates (x, y) into $(\tilde{x}, \tilde{y})$ by assuming $\tilde{x} = x - u(y)$, $\tilde{y} = y$, where $u(y)$ is a function such that $u'(y) = w(y)$. Then (36) becomes

$$\begin{aligned} &\partial_{\tilde{x}} h(\tilde{x}, \tilde{y}) = w(\tilde{y}) + \tilde{b}(\tilde{x} + u(\tilde{y}), \tilde{y}), \\ &\partial_{\tilde{y}} h(\tilde{x}, \tilde{y}) - w(\tilde{y})\partial_{\tilde{x}} h(\tilde{x}, \tilde{y}) = -w(\tilde{y})\tilde{b}(\tilde{x} + u(\tilde{y}), \tilde{y}). \end{aligned} \tag{37}$$

Hence, we deduce that $\partial_{\tilde{y}} h(\tilde{x}, \tilde{y}) = (w(\tilde{y}))^2$ and consequently the function h can expressed as a sum $h(\tilde{x}, \tilde{y}) = h_1(\tilde{x}) + h_2(\tilde{y})$ in a neighborhood of p; and therefore $\partial_{\tilde{x}} h(\tilde{x}, \tilde{y}) = \partial_{\tilde{x}} h_1(\tilde{x})$. If we suppose $v(\tilde{x}) = \partial_{\tilde{x}} h_1(\tilde{x})$, then from (37), we deduce $\tilde{b}(\tilde{x} + u(\tilde{y}), \tilde{y}) = v(\tilde{x}) - u'(\tilde{y})$. Finally, considering the above and returning to the coordinates (x, y), we obtain (32) and (33).

Conversely, when using (29), (32) and (33), it is straightforward to verify that (34) holds, which by virtue of Proposition 9 completes the proof. □

Corollary 11. *Let M be a 3-dimensional essentially weakly para-cosymplectic manifold whose curvature does not vanish at any point. Assume that M admits a non-zero Killing vector field K.*

(i) *If L is a Killing vector field on M, then $L = cK$, $c =$ const.*
(ii) *The scalar curvature of M is constant along any trajectory of K.*

Proof. (i) Let L and K be two non-zero Killing vector fields on M. Then in a canonical chart, $K_y =$ const. $\neq 0$, $L_y =$ const. $\neq 0$ and $N = (1/K_y)K - (1/L_y)L$ is also a Killing vector field on M. But $N_y = 0$ implies immediately $N = 0$, so $L = (L_y/K_y)K$.

(ii) By (30) and (32), in a canonical chart, we have $r = 2v''(x - u(y))$. With the help of (33), we get now $Kr = 0$. □

5. Local homogeneity

Theorem 12. *Let M be a 3-dimensional essentially weakly para-cosymplectic manifold, whose curvature is non-zero at any point. Then M is irreducible as a Riemannian manifold.*

Proof. Let us assume that the assertion does not hold. Let M satisfy the assumptions and be locally reducible around some point $p \in M$. Then, because of the dimension, there exists a non-zero, non-isotropic, parallel vector field K on a sufficiently small neighborhood $\mathcal{O}$ of p. By $\nabla K = 0$, we have $R(X, Y)K = 0$, which by virtue of (17) implies $r\Phi(X, Y)\varphi K = 0$. Since $r \neq 0$ (cf. (17)), then $\varphi K = 0$ and hence $K = a\xi$ for a certain function a. Differentiating the last equation covariantly and projecting the result obtained onto ξ, we get $0 = \mathrm{d}a$. Consequently, a is a non-zero constant. Therefore, $A = -\nabla\xi = -(1/a)\nabla K = 0$, which with the help of (5) leads to $\nabla\varphi = 0$. But this is a contradiction. □

Theorem 13. *Let M be 3-dimensional weakly para-cosymplectic manifold. If M is locally homogeneous as a Riemannian manifold, then it is para-cosymplectic or locally flat.*

Proof. Let us suppose that M is not locally flat. By the homogeneity, $R \neq 0$ at every point of M. Consider the set Z of points at which $\nabla\varphi \neq 0$. We will show that Z is empty. Assume on the contrary that Z is not empty and fix a point $p \in Z$. We consider a neighborhood $\mathcal{O}$ of p, on which there is defined the structure frame (V, U, ξ) like in Lemma 2. Let q be an arbitrary point of $\mathcal{O}$. By virtue of Proposition 8, for an isometry f such that $f(p) = q$, we have $\varepsilon\beta(p) = (\beta \circ f)(p) = \beta(q)$. Consequently, $\beta = \text{const.}$ on $\mathcal{O}$. Now, by (18) and (17), $R = 0$ on $\mathcal{O}$, which is a contradiction. □

Before we classify all 3-dimensional locally homogeneous weakly para-cosymplectic manifolds, we describe three typical examples of such manifolds.

Example 14. Let (N, J, G) be a homogeneous 2-dimensional para-Kählerian manifold (for such manifolds, see [6]). Then the structure (φ, ξ, η, g) defined on the product manifold $M = N \times \mathbb{R}$ by

$$\varphi = (J, 0), \quad \xi = \partial_t, \quad \eta = \mathrm{d}t, \quad g = G \times \mathrm{d}t^2,$$

t being the Cartesian coordinate on $\mathbb{R}$, is homogeneous and para-cosymplectic.

Example 15. Let $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. Let $\mathbb{H}_3$ denote the matrix group consisting of matrices of the form

$$\begin{bmatrix} 1 & \varepsilon_1 z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

where $(x, y, z) \in \mathbb{R}^3$. Then, $\mathbb{H}_3$ is isomorphic to the Heisenberg group and its underlying manifold is $\mathbb{R}^3$. The vector fields

$$V = \partial_x, \quad U = \varepsilon_1 z\partial_x + \partial_y, \quad \xi = \partial_z$$

are left-invariant and form a basis of the Lie algebra of $\mathbb{H}_3$. Define a left-invariant almost para-contact hyperbolic metric structure on $\mathbb{H}_3$ by

$$\varphi V = \varepsilon_2 V, \quad \varphi U = -\varepsilon_2 U, \quad \varphi\xi = 0, \quad \eta(\cdot) = g(\cdot, \xi), \quad g(\xi, \xi) = g(V, U) = 1.$$

By Proposition 5, $\mathbb{H}_3$ endowed with this structure becomes an essentially weakly para-cosymplectic manifold. For this manifold, we have $\beta = 0$, so it is flat.

Example 16. Let $\beta = \text{const.} \neq 0$ and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. Let $\mathbb{A}_3(\beta)$ be the matrix group consisting of matrices of the form

$$\begin{bmatrix} e^{-y} & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & e^{z} \end{bmatrix},$$

where $(x, y, z) \in \mathbb{R}^3$. Then $\mathbb{A}_3(\beta)$ is a solvable, non-unimodular Lie group (which is isomorphic to the product of a 2-dimensional non-abelian Lie group and $\mathbb{R}$) and its underlying manifold is $\mathbb{R}^3$. The vector fields

$$V = \partial_x, \quad U = -\beta x \partial_x + \beta \partial_y, \quad \xi = -\varepsilon_1 \beta^{-1} \partial_x + \partial_z$$

are left-invariant and form a basis of the Lie algebra of $\mathbb{A}_3(\beta)$. Define a left-invariant almost para-contact hyperbolic metric structure on $\mathbb{A}_3(\beta)$ by

$$\varphi V = \varepsilon_2 V, \quad \varphi U = -\varepsilon_2 U, \quad \eta(\cdot) = g(\cdot, \xi), \quad g(\xi, \xi) = g(V, U) = 1.$$

By Proposition 5, $\mathbb{A}_3(\beta)$ endowed with this structure becomes an essentially weakly para-cosymplectic manifold. For this manifold, we have $\beta = \text{const.} \neq 0$, so it is also flat. But obviously this structure differs from those defined in the previous examples.

Theorem 17. *Let M be a 3-dimensional weakly para-cosymplectic manifold, which is locally homogeneous as a Riemannian manifold. Then M is locally isomorphic to*

(i) *a product of a 2-dimensional homogeneous para-Kähler manifold and an open interval in the case when $\nabla\varphi = 0$; or*
(ii) *$\mathbb{H}_3$ endowed with the essentially weakly para-cosymplectic structure as in Example 15 when $\nabla\varphi \neq 0$ and $\beta = 0$; or*
(iii) *$\mathbb{A}_3(\beta)$ endowed with the essentially weakly para-cosymplectic structure as in Example 16 when $\nabla\varphi \neq 0$ and $\beta = \text{const.} \neq 0$.*

Proof. If M is para-cosymplectic, then it is locally a product of a 2-dimensional para-Kähler manifold and an open interval; and next by the homogeneity the product is like in Example 14. Let M be non-para-cosymplectic. By Theorem 13, it is locally flat, and by Proposition 8, the function β is constant. We have two possibilities: $\beta = 0$ and $\beta \neq 0$. Now using Proposition 5, we see that locally the structure (φ, ξ, η, g) can be described like in Examples 15 and 16. □

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